TRINITY COLLEGE

Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section One: Calculator-free

SOLUTIONS

Student number: In figures | | | | | | | | |

In words _____

Your name _____

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	53	35
Section Two: Calculator-assumed	13	13	100	97	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

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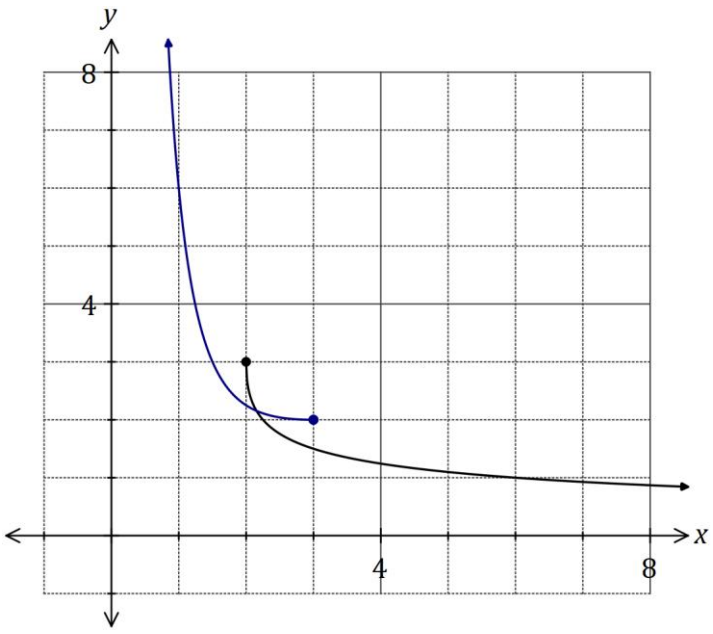
Section One: Calculator-free **35% (53 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

The graph of $y = f(x)$ is shown below, where f is defined by $f(x) = \dfrac{3}{1+\sqrt{x-2}}$.



(a) Sketch the graph of $y = f^{-1}(x)$ on the same axes. (2 marks)

Solution
See graph
Specific behaviours
✓ reflection in $y = x$
✓ starts at $(3,2)$ and through $\approx (2.2, 2.2)$

(b) Determine the defining rule for $y = f^{-1}(x)$ and state its domain. (4 marks)

Solution
$1 + \sqrt{y-2} = \dfrac{3}{x}$ $y = f^{-1}(x) = \left(\dfrac{3}{x} - 1\right)^2 + 2$ $D_{f^{-1}}: \ 0 < x \leq 3$
Specific behaviours
✓ inverts variables and starts to rearrange
✓ correct inverse
✓ correct lower bound of domain
✓ correct upper bound of domain

See next page

Question 2 **(7 marks)**

Consider $f(z) = 5z^3 + 2z^2 + 10z + 4$, where z is a complex number.

(a) Determine, with reasons, which of the following are factors of $f(z)$.

(i) $z - 2$. (2 marks)

Solution
$f(2) = 40 + 8 + 20 + 4 = 72$ $z - 2$ is NOT a factor since $f(2) \neq 0$
Specific behaviours
✓ evaluates $f(2)$ with evidence ✓ reason

(ii) $z - \sqrt{2}i$. (2 marks)

Solution
$f(\sqrt{2}i) = 5 \times 2\sqrt{2}i^3 + 2 \times 2i^2 + 10\sqrt{2}i + 4$ $= -10\sqrt{2}i - 4 + 10\sqrt{2}i + 4$ $= 0$ $z - \sqrt{2}i$ is a factor since $f(\sqrt{2}i) = 0$
Specific behaviours
✓ evaluates $f(\sqrt{2}i)$ with evidence ✓ reason

(b) Solve the equation $f(z) = 0$. (3 marks)

Solution
Second factor is $z + \sqrt{2}i$ $f(z) = (z - \sqrt{2}i)(z + \sqrt{2}i)(az + b)$ $= (z^2 + 2)(az + b)$ $= (z^2 + 2)(5z + 2)$ $f(z) = 0 \Rightarrow z = \pm\sqrt{2}i, \quad z = -\frac{2}{5}$
Specific behaviours
✓ indicates conjugate factor ✓ factorises $f(z)$ ✓ all three solutions

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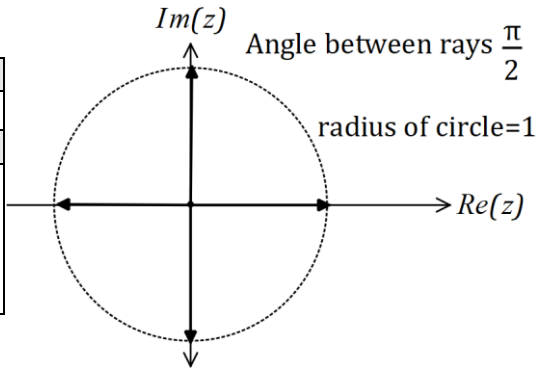
Question 3 **(7 marks)**

(a) Locate the roots of the complex equation $z^4 - 1 = 0$ in the Argand plane below. (3 marks)

Solution
See graph
Specific behaviours
✓ 4 evenly spaced rays
✓ indicates ray spacing
✓ indicates ray length
(May use rays or points)



(b) State the sum of all the roots of the complex equation $z^4 - 1 = 0$. (1 mark)

Solution
$\text{Sum} = 0$
Specific behaviours
✓ states sum

(c) Let u be any 4^{th} root of unity, where $\text{Im}\, u \neq 0$.

Show that $u(1 + 5u)(1 + 5u^2) = 4(5 - u)$. (3 marks)

Solution
$(u + 5u^2)(1 + 5u^2) = u + 5u^2 + 5u^3 + 25u^4$ $= 5(1 + u + u^2 + u^3) - 5 - 4u + 25$ $= 5(0) + 20 - 4u$ $= 4(5 - u)$
Specific behaviours
✓ expands
✓ uses $u^4 = 1$
✓ uses sum of roots and simplifies

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Question 4 **(6 marks)**

(a) Solve this system of equations. (3 marks)

$$x + y + 2z = 1$$
$$4x + y - z = 7$$
$$3x - y + z = 14$$

Solution
$Eq(2) + Eq(3) \Rightarrow 7x = 21 \Rightarrow x = 3$ $Eq(1) + Eq(3) \Rightarrow 4(3) + 3z = 15 \Rightarrow z = 1$ $3 + y + 2(1) = 1 \Rightarrow y = -4$ $x = 3, \quad y = -4, \quad z = 1$
Specific behaviours
✓ eliminates one variable using appropriate technique ✓ value of one variable ✓ values of second and third variables

(b) Determine the value of constant a so that the following system of equations does not have a unique solution and give a brief geometric interpretation of the system of equations with this value. (3 marks)

$$x + y + 2z = 1$$
$$4x + y - z = 7$$
$$ax - y + z = 14$$

Solution
$Eq(2) + Eq(3) \Rightarrow (4 + a)x = 21$ $a = -4 \ (\text{Since } 0x = 21 \text{ impossible})$ Two parallel planes cut by the third non-parallel plane.
Specific behaviours
✓ adds second and third equations ✓ states value of a ✓ indicates parallel planes

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Question 5 **(10 marks)**

The points A, B and C have position vectors $(1, 0, -2)$, $(b, -2, 1)$ and $(2, -1, 0)$ respectively.

(a) Determine the vector equation for the line through A and C. (2 marks)

Solution
$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (1, -1, 2)$ $\mathbf{r} = (1, 0, -2) + \lambda(1, -1, 2)$
Specific behaviours
✓ direction of line ✓ vector equation

(b) Determine, in terms of b, the Cartesian equation of the plane containing A, B and C. (5 marks)

Solution
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (b - 1, -2, 3)$ $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} b-1 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5-2b \\ 3-b \end{pmatrix}$ $\mathbf{n} \cdot \overrightarrow{OA} = -1 + 2b - 6 = 2b - 7$ $-x + (5 - 2b)y + (3 - b)z = 2b - 7$
Specific behaviours
✓ second vector in plane ✓ uses cross product ✓ normal vector ✓ constant ✓ Cartesian equation

(c) The line with equation $\mathbf{r} = (3, -5, 6) + \mu(2, q, -12)$ is perpendicular to the plane containing A, B and C. Determine the values of the constants b and q. (3 marks)

Solution
$(2, q, -12) = k(-1, 5 - 2b, 3 - b) \Rightarrow k = -2$ $-12 = -2(3 - b) \Rightarrow b = -3$ $q = -2\big(5 - 2(-3)\big) = -22$
Specific behaviours
✓ indicates normal and line direction parallel ✓ value of b ✓ value of q

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Question 6 **(5 marks)**

The complex numbers u and v satisfy the equations $u - v = 2i$ and $uv = 10$.

Solve the equations for u and v, giving your solution(s) in the form $x + yi$, where x and y are real.

Solution
$u = v + 2i \Rightarrow v(v + 2i) = 10$ $v^2 + 2iv = 10$ $(v + i)^2 - i^2 = 10$ $(v + i)^2 = 9$ $v = \pm 3 - i$ $v = 3 - i$, $u = 3 + i$ or $v = -3 - i$, $u = -3 + i$
Specific behaviours
✓ eliminates u or v to form quadratic ✓ completes square ✓ solves quadratic correctly ✓ states one pair of solutions ✓ states second pairs of solutions

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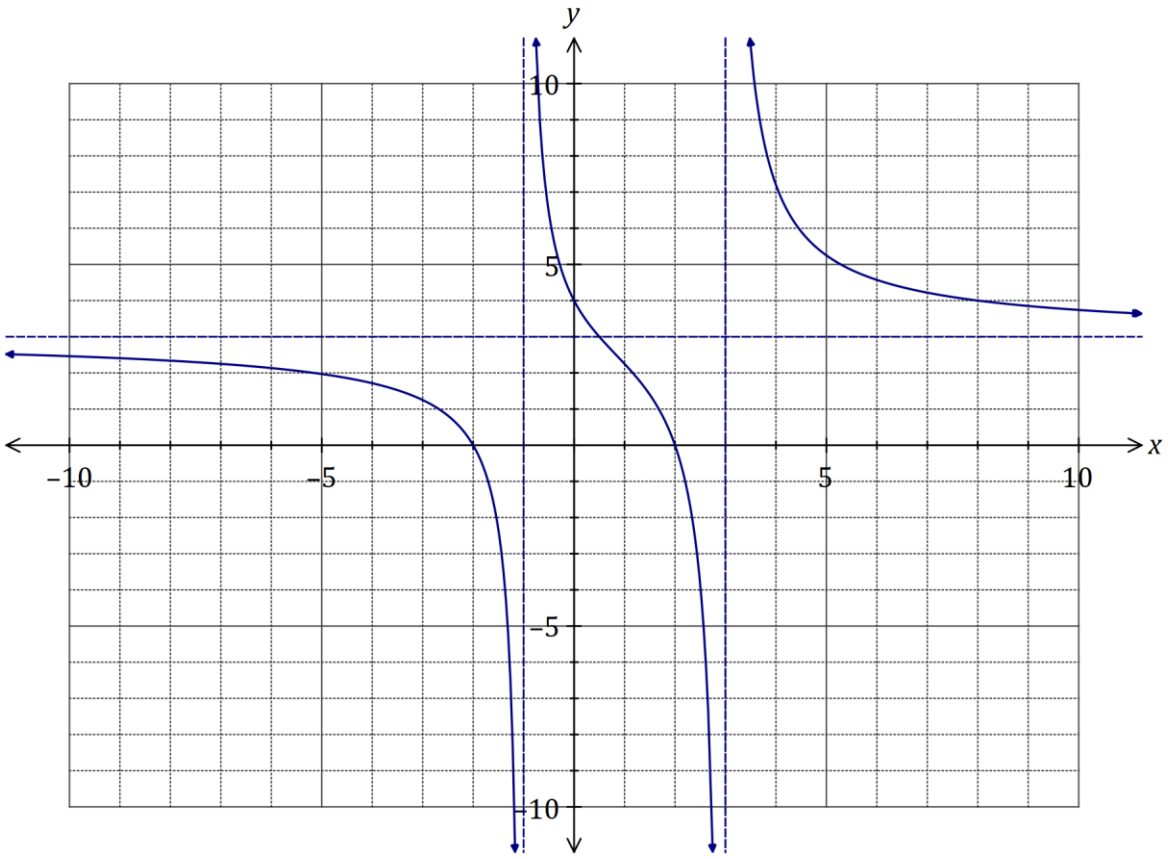
Question 7 **(6 marks)**

The graph of $y = \dfrac{3x^2 - 12}{(x+1)(x-3)}$ has no stationary points. Sketch the graph.



Solution
See graph
Specific behaviours
✓ indicates vertical asymptotes $x = -1, x = 3$
✓ indicates horizontal asymptote $y = 3$
✓ through $(-2, 0)$ and correct curvature for $x < -1$
✓ through $(2, 0)$ and $(0, 4)$
✓ correct curvature for $-1 < x < 3$
✓ close to $(5, 5)$ and correct curvature for $x > 3$

See next page

Question 8 **(6 marks)**

A function is defined by $f(x) = \dfrac{x+4}{(3x+2)(5x-4)}$.

(a) State the natural domain of $f(x)$. (1 mark)

Solution
$\left\{x: x \in \mathbb{R}, x \neq -\frac{2}{3}, x \neq \frac{4}{5}\right\}$
Specific behaviours
✓ indicates exceptions

(b) State the equations of all asymptotes of the graph of $y = x \cdot f(x)$. (2 marks)

Solution
$x \cdot f(x) = \dfrac{x(x+4)}{(3x+2)(5x-4)}$ Vertical: $x = -\dfrac{2}{5}, x = \dfrac{4}{5}$ Horizontal: $x \to \infty, x \cdot f(x) \to \dfrac{x^2}{15x^2} \to \dfrac{1}{15} \Rightarrow y = \dfrac{1}{15}$
Specific behaviours
✓ both vertical asymptotes ✓ horizontal asymptote *(penalise if not given as equation)*

(c) The graph of $y = \dfrac{1}{f(x)}$ has an asymptote with equation $y = ax + b$. Determine the values of the constants a and b. (3 marks)

Solution
$\dfrac{1}{f(x)} = \dfrac{(3x+2)(5x-4)}{x+4}$ $= \dfrac{15x^2 - 2x - 8}{x+4}$ $= 15x - 62 - \dfrac{240}{x+4}$ $a = 15, b = -62$
Specific behaviours
✓ correct expansion of numerator and denominator ✓ value of a ✓ value of b

End of questions

Supplementary page

Question number: _____